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Mathematical Methods

2006

Trial Examination 2

SECTION 1 Multiple-choice questions

Instructions for Section 1

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1
The solution(s) to the equation $e^{2x+2} = e^{x}$ is/are

A. $-1, -2$
B. $1, 2$
C. $1, -2$
D. -1
E. -2

Question 2
The **exact** numerical value of the sum of the first two negative and the first two positive solutions of the equation $3\cos(7x)+1=0$

A. does not exist
B. cannot be determined
C. $\frac{2}{7}\left[\cos^{-1}\left(-\frac{1}{3}\right)+\pi\right]$
D. $\frac{2\pi}{7}$
E. 0

Question 3
$\log_2\left(4a^{p}\right)$ is equal to

A. $\frac{p\log_e(4a)}{\log_e 2}$
B. $p\left(2^{a}\right)$
C. $\frac{2}{p\log_a 2}$
D. $2+\frac{p}{\log_a 2}$
E. $2ap$

Question 4
$f(x)=\log_{10}(x+1)^2$ is defined for x in

A. R B. $R\setminus\{0\}$ C. $[-1,\infty)$ D. $(-\infty,-1)$ E. $[-2,1)$

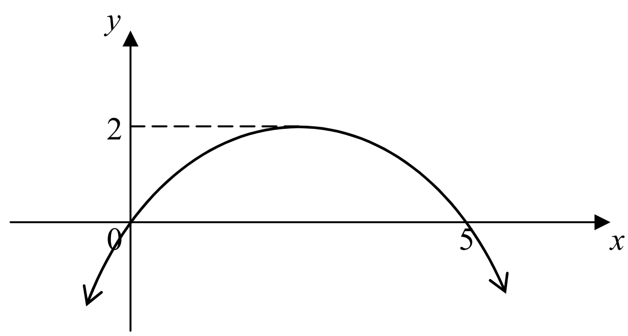
Question 5
The set of values of x for which $|2x-1|<1$ is

A. $0 \le x \le 1$
B. $-\frac{1}{2} < x < \frac{1}{2}$
C. $-\frac{1}{2} \le x \le \frac{1}{2}$
D. $x < -\frac{1}{2}$ or $x > \frac{1}{2}$
E. $0 < x < 0.9$

Question 6
The function $f(x) = -x^3 + bx^2 + cx + d$ has three x-intercepts. The x-coordinate of the local maximum is between the two adjacent x-intercepts, -3 and 1. The other x-intercept is p where

A. $p < -3$
B. $p < 1$
C. $p > -3$
D. $p > 1$
E. $-3 < p < 1$

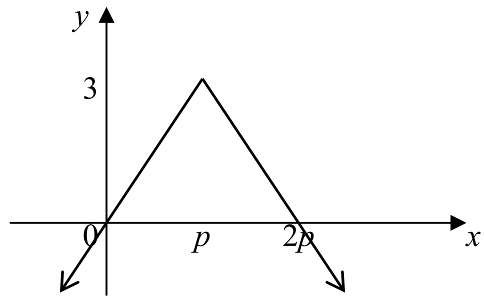
Question 7
The graph of $y = a(x+b)^2 + c$ is shown below.



The values of the parameters could be

A. $a = -\frac{4}{5}$, $b = -\frac{5}{2}$, $c = -2$
B. $a = -\frac{8}{25}$, $b = \frac{5}{2}$, $c = -2$
C. $a = -\frac{4}{5}$, $b = -\frac{5}{2}$, $c = 2$
D. $a = -\frac{8}{25}$, $b = -\frac{5}{2}$, $c = 2$
E. $a = -\frac{8}{25}$, $b = \frac{5}{2}$, $c = 2$

Question 8



The equation of the graph shown could be

A. $y = 3\left(\frac{|p-x|}{p} + 1\right)$

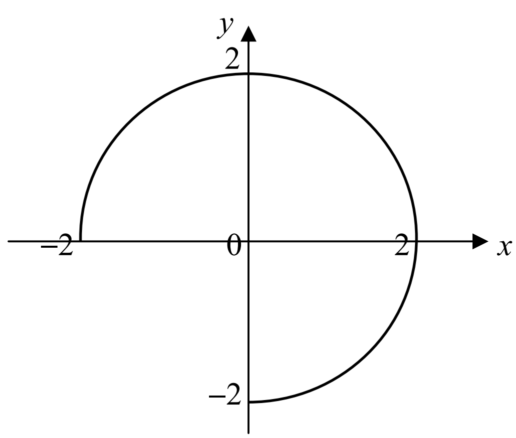
B. $y = 3\left(\frac{|p-x|}{p} - 1\right)$

C. $y = 3\left(1 - \frac{|p-x|}{p}\right)$

D. $y = 3\left(\frac{|x-p|}{p} + 1\right)$

E. $y = 3\left(\frac{|-x-p|}{p} + 1\right)$

Question 9



Referring to the graph shown, which one of the following statements is **false**?

A. The relation does not have an inverse.
B. The relation is not a function.
C. The relation is not a one-to-one function.
D. The inverse of the relation is not a function.
E. The inverse of the relation is the relation.

Question 10
The graph of $f(x)=-x^2+x$ undergoes the following transformations in the order as shown below.
Translation to the left by $\frac{1}{2}$.
Downward translation by $\frac{1}{4}$.
Reflection in the x-axis.

The rule for the resulting graph is

A. $g(x)=x^2$
B. $g(x)=x^2-\frac{1}{2}$
C. $g(x)=\left(x-\frac{1}{2}\right)^2+\frac{1}{4}$
D. $g(x)=x^2+\frac{1}{2}$
E. $g(x)=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}$

Question 11
The equation $x+\sin\left(\frac{\pi x}{2}\right)-c=0$ will have more than one solution in $[0,4]$ provided

A. $1.7<c<1.8$
B. $2.2<c<2.3$
C. $1.8<c<2.2$
D. $c>1.8$
E. $c<2.2$

Question 12
The growth of a population is given by $N(t)=5\times 2^{0.1t}$, where N is in thousands and t is the number of years after 1 Jan 2000.
The rate of growth of the population (in thousands per year) on 1 Jan 2010 is closest to

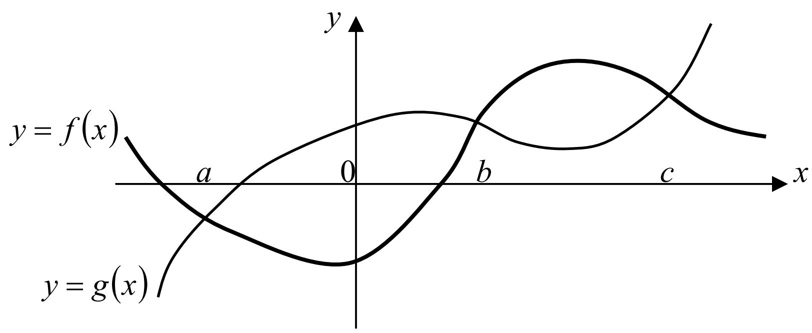
A. 0.7
B. 1
C. 7
D. 10
E. 1000

Question 13

The depth of water is given by $h(t)=1.5+0.6\cos\left(\frac{\pi}{6}t\right)$ for $0\le t\le 12$, where h is in metres and t is the number of hours after midnight.
The average rate of change in the depth of water (in metres per hour) between 6.00 am and 8.00 am is

A. -0.157
B. 0.150
C. -0.040
D. 0.040
E. 0.157

Question 14



The total area, bounded by the two curves $y=f(x)$ and $y=g(x)$ on the interval (a,c), is given by

A. $\int_a^c (f(x)-g(x))dx$
B. $\int_a^c (g(x)-f(x))dx$
C. $\int_a^b (f(x)-g(x))dx+\int_b^c (f(x)-g(x))dx$
D. $\int_b^a (f(x)-g(x))dx+\int_b^c (f(x)-g(x))dx$
E. $\int_a^b (f(x)-g(x))dx-\int_b^c (f(x)-g(x))dx$

Question 15

If $F(x)=\int f(x)dx$, then $\int_0^1 2(x-f(x))dx$ is equal to

A. $1-2F(1)+2F(0)$
B. $1-2F(1)-2F(0)$
C. $1-2F(1)$
D. $2(1-F(1)+F(0))$
E. $2(1-F(1)-F(0))$

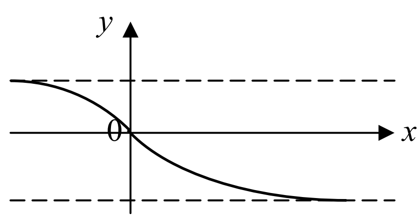
Question 16
If $f(x)=\left|\cos\left(\frac{x}{2}\right)\right|$ and $\pi < a < 3\pi$, then $f'(a)$ is equal to

A. $-2\cos\left(\frac{a}{2}\right)$
B. $\frac{1}{2}\cos\left(\frac{a}{2}\right)$
C. $-2\sin\left(\frac{a}{2}\right)$
D. $\frac{1}{2}\sin\left(\frac{a}{2}\right)$
E. $-\left|\frac{1}{2}\sin\left(\frac{a}{2}\right)\right|$

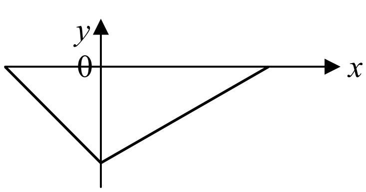
Question 17
Using the approximation $f(a+h)\approx f(a)+hf'(a)$, the value of $\sqrt{15}$, correct to three decimal places, is

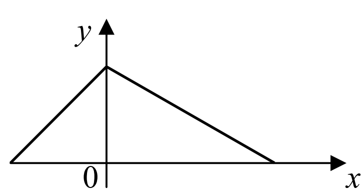
A. 3.872
B. 3.873
C. 3.875
D. 3.876
E. 4.000

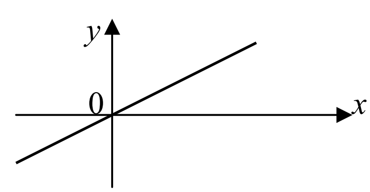
Question 18
The graph of $y=f(x)$ is shown below.

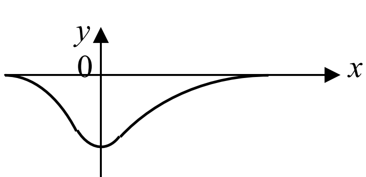


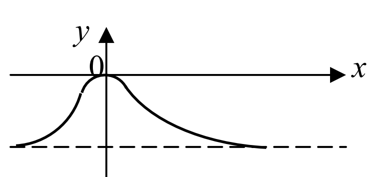
Which one of the following could be the graph of $y=f'(x)$?

A.


B.


C.


D.


E.


Question 19
A positively skewed binomial distribution with parameters n (number of trials) and p (probability of success in a single trial) will certainly tend towards bell-shaped if

A. n is smaller
B. p is smaller
C. n is greater
D. p is greater
E. $p = 0$ or 1

Question 20
The number, X, of televisions in a family is a random variable with the following probability distribution.

x	0	1	2	3
$\Pr(X = x)$	0.1	0.3	0.4	0.2

If a family with at least one television is selected at random, the probability that the family has no more than two televisions is

A. $\frac{7}{10}$ B. $\frac{7}{9}$ C. $\frac{7}{8}$ D. $\frac{4}{5}$ E. $\frac{1}{2}$

Question 21
The random variable X has a normal distribution with mean μ and standard deviation 4.
If Z has the standard normal distribution, then

A. $\Pr(X < \mu + 4) = \Pr(Z < 2)$
B. $\Pr(X < \mu + 4) = 1 - \Pr(Z < 2)$
C. $\Pr(X < \mu + 8) = 1 - \Pr(Z < 2)$
D. $\Pr(X > \mu + 8) = 1 - \Pr(Z < 2)$
E. $\Pr(X > \mu + 8) = \Pr(Z < 2)$

Question 22
The random variable X has a probability density function given by

$$f(x) = \begin{cases} k\sin(\pi x) & if \quad 1 < x < 2 \\ 0 & elsewhere \end{cases}$$

The value of k is

A. $\frac{1}{2}$ B. $-\frac{1}{2}$ C. $\frac{\pi}{2}$ D. $-\frac{\pi}{2}$ E. π

SECTION 2 Extended-answer questions

Instructions for Section 2

Answer **all** questions.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.

Question 1

Consider the function $f(x) = x^3 - 6x^2 + 12x + p$, where p is a real constant.

a. Find the values of b and c in terms of p if necessary such that $f(x) = (x+b)^3 + c$. 3 marks

b. Hence show that $x^3 - 6x^2 + 12x + p = 0$ has only one real solution for all real p. 2 marks

c. i. Find the value of p such that $f(x) = x^3 - 6x^2 + 12x + p$ has a stationary point on the x-axis. 1 mark

ii. Hence sketch the graph of $y = |f(x)|$. Label the axis intercepts clearly. 2 marks

d. If $f(x+u)+v=x^3$, find the values of u and v in terms of p. 2 marks

e. **i.** Find $f^{-1}(x)$ for $p=-7$. 2 marks

ii. Find the derivative of $f^{-1}(x)$. State the maximal domain of the derivative. 3 marks

f. Given $g(x)=(x-\alpha)^3+\alpha$ and $\alpha \in R$, find the area of the regions enclosed by the graphs of $y=g(x)$ and $y=g^{-1}(x)$. 3 marks

Total 18 marks

Question 2

A particular radioactive substance (the parent substance) decays into a stable substance (the daughter substance). The amount of the parent substance remaining after t years is given by

$$P(t) = Ae^{-at}\text{, where } A \text{ and } a \text{ are positive real constants.}$$

a. Show that the initial amount of the parent substance is A. 1 mark

b. **i.** Show that the activity, i.e. the rate of decay of the particular radioactive substance is directly proportional to the amount of the substance remaining.

2 marks

ii. Determine the time (in terms of a) required for the activity to be halved.

2 marks

c. **i.** In terms of $P(t)$ and A, write an expression for $D(t)$, the amount of the daughter substance after t years. 1 mark

ii. Hence show that $t = \frac{1}{a}\log_e\left(\frac{D(t)}{P(t)} + 1\right)$. 2 marks

d. **i.** A sample of the particular radioactive substance is analysed. The ratio of the amount of daughter substance to the amount of parent substance, i.e. $\frac{D(t)}{P(t)}$, is found to be 0.0196. Given that $a = 1.39 \times 10^{-11}$, determine the age (in years) of the sample to three significant figures.

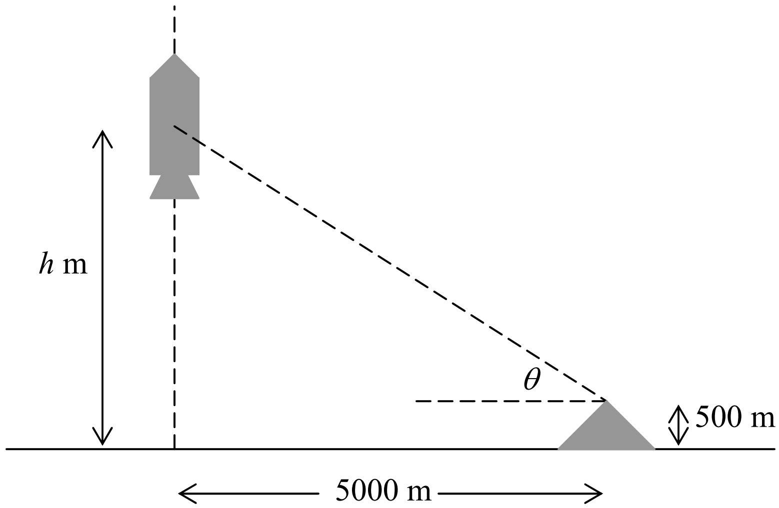
2 marks

ii. **Use the approximation** $\Delta y \approx \frac{dy}{dx}\Delta x$ to find the number of years (to three significant figures) by which the age of the sample is overestimated if the ratio $\frac{D(t)}{P(t)}$ was overestimated by 0.00130.

3 marks
Total 13 marks

Question 3

An observer on top of a 500-m hill watches a rocket blast off vertically 5000 metres away. At time t seconds after blast off, the altitude of the rocket is h metres and the line of sight from the observer to the rocket makes an angle of θ radians with the horizontal. θ is positive for elevation, negative for depression.



a. **i.** Determine the lower bound for θ to three decimal places. 1 mark

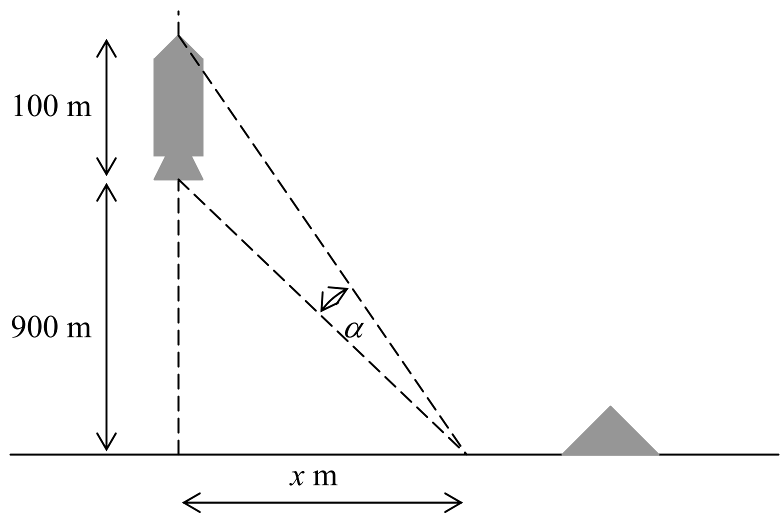
ii. Express h in terms of θ. 1 mark

b. Show that $\dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{5000} \times \dfrac{dh}{dt}$. 3 marks

c. If $\dfrac{dh}{dt}$ is constant, find θ at which maximum $\dfrac{d\theta}{dt}$ occurs. 3 marks

(Do not use calculators)

Another observer watches the rocket blast off vertically x metres away. The length of the rocket is 100 m, and the following diagram shows the moment when it is at an altitude of 900 m. The observer has the best view when α is maximum.



d. i. Show that $\alpha = \tan^{-1}\left(\frac{1000}{x}\right) - \tan^{-1}\left(\frac{900}{x}\right)$. 2 marks

ii. How far away from the launch site would the observer be in order to get the best view of the rocket at that particular moment? (Answer to the nearest metre)

2 marks

iii. What is the best viewing angle (in degrees to two decimal places) at that particular moment?

1 mark

iv. What is the best viewing angle (in degrees to two decimal places) at that particular moment if the observer must be at least 2000m from the launch site?

1 mark

Total 14 marks

Question 4

Nails produced by a machine do not have uniform length L and diameter d. The random variable L has a normal distribution with $\mu = 5.00$ cm and $\sigma = 0.02$ cm. d in mm is also a random variable, it has a probability density function

$$f(d) = \begin{cases} 750(d-3.9)(4.1-d) & if \quad 3.9 \le d \le 4.1 \\ 0 & otherwise \end{cases}$$

A nail produced by the machine is considered to be unacceptable if its length L is outside the interval [4.95, 5.05] and its diameter d is outside [3.92, 4.08]. Variations in L and d are independent.

a. What is the probability (correct to 3 decimal places) that a randomly selected nail produced by the machine has an acceptable length?

1 mark

b. What is the probability (correct to 3 decimal places) that a randomly selected nail produced by the machine has an acceptable diameter?

2 marks

c. What proportion (correct to 3 decimal places) of the nails produced by the machine is unacceptable?

2 marks

d. What proportion (correct to 3 decimal places) of the unacceptable nails has acceptable length?

2 marks

e. Twenty nails are selected randomly for inspection. What is the probability that at least 95% of them are acceptable?

3 marks

f. The probability of a second inspection of another twenty randomly selected nails is 0.9 if less than 95% of the nails in the first inspection are acceptable; the probability of a second inspection of another twenty randomly selected nails is 0.3 if at least 95% of the nails in the first inspection are acceptable. What is the probability that a second inspection is not carried out?

3 marks

Total 13 marks

End of exam 2

